Math 249 Lecture 4 Notes

Daniel Raban

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1 Finite Group Representations over $\mathbb{C}$

1.1 Permutation representations

Definition 1.1. A *matrix representation* is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$.

Example 1.1. The symmetric group S_n has a matrix representation $S_n \to \mathrm{GL}_n(\mathbb{C})$ sending $\sigma \mapsto P_\sigma$, where $(P_\sigma)_{i,j} = \delta_{i,\sigma(j)}$ (the corresponding permutation matrix). Note that $P_\sigma e_j = e_{\sigma(j)}$.

Definition 1.2. For $\sigma \in S_n$, the *sign* $(-1)^\sigma$ of σ is $\det(P_\sigma)$.

The sign function is a homomorphism from $S_n \to \{\pm 1\}$. Also, any transposition (a permutation of the form $\sigma = (i\ j)$) has sign -1. Since any permutation can be factored into a product of transpositions, this provides a method of computing the sign of a permutation. In particular, a k-cycle is the product of $k-1$ transpositions; $(i_1 \ \cdots \ i_{k-1}\, i_k) = (i_1 i_k)(i_1 \ \cdots \ i_{k-1})$. Then we can also say that the sign of σ is $(-1)^{\text{number of even length cycles}}$.

Definition 1.3. The *signed permutation group* B_n is the set of matrices in $\mathrm{GL}_n(\mathbb{C})$ with entries either 0 or ± 1 and with only 1 nonzero entry in each row and column.

We have a short exact sequence

$$1 \to \{\pm 1\}^n \to B_n \to S_n \to 1.$$

In fact, $B_n = S_n \rtimes \{\pm 1\}^n$, where the action of S_n on $\{\pm 1\}^n$ permutes the components.

1.2 Reflections in $\mathbb{R}^n$

Definition 1.4. A *reflection* in $\mathbb{R}^n$ is $T : \mathbb{R}^n \to \mathbb{R}^n$ such that T is diagonalizable with entries 1 (multiplicity $n-1$) and -1 (with multiplicity 1).

This makes $\mathbb{R}^n = \underbrace{V}_{\dim n-1} \oplus \underbrace{\mathbb{R}v}_{\dim 1}$, where $T|_V = \mathrm{id}$ and $T(v) = -v$.

In S_n, the image of a transposition $\tau = (i\,j)$ under the representation is a reflection P_τ. This is because

$$P_\tau e_k = e_k \text{ for } k \neq i, j \qquad P_\tau(e_i + e_j) = e_i + e_j \qquad P_\tau(e_i - e_j) = -(e_i - e_j).$$

In B_n, the reflections are

1. $\tau = (i\,j) \in S_n$
2. elements of the form $(i\,\bar{j})_+$, which send $e_i \mapsto -e_j$ and $e_j \mapsto -e_i$
3. elements of the form $(i)_-$, which send $e_i \mapsto -e_i$.

Let's clarify the cycle notation for B_n. The bar over the j refers to -1 times the jth element. The $+$ subscript means that $\bar{j} \mapsto i$; a $-$ sign would mean that $j \mapsto -i$.

1.3 More basic definitions

Definition 1.5. Matrix representations ρ and ρ' are *similar* if there is some $S \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho'(g) = S\rho(g)S^{-1}$.

This basically amounts to the matrices being expressed in different representations. If two representations ρ and ρ' are similar, then $\rho(g)$ and $\rho'(g)$ will be similar matrices.

Definition 1.6. The *character* of a representation ρ is the function $\chi_\rho(g) = \mathrm{tr}(\rho(g))$.

Characters are constant on conjugacy classes because

$$\chi_\rho(hgh^{-1}) = \mathrm{tr}(\rho(hgh^{-1})) = \mathrm{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{tr}(\rho(g)) = \chi_\rho(g),$$

using the fact that the trace is invariant under conjugation. So if ρ and ρ' are similar, then $\chi_{\rho'} = \chi_\rho$. So χ_ρ determines ρ up to similarity!

It is often useful to work in a little more generality:

Definition 1.7. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $\mathrm{GL}(V)$ be the set of invertible linear maps $V \to V$. A *linear representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Remark 1.1. If $V = \mathbb{C}^n$, then the linear representation is the same as the matrix representation. If $V \cong \mathbb{C}^n$, then by choosing a basis, we can naturally recover the matrix representation from the linear representation. Choosing a different basis will result in a similar representation.